Yiddish word of the day

"yasher koakh"
(shkoyakh) = יישר כּח

= Congratulation

Yiddish phrase of the day

"di verem zoln
dikh esn" = די ווערעם זאָלן
דיך עסן

= may the worms eat
you!

Eigenvalues For Linear Operators

· **Reminder to fill out EVALS !!!**

- We are currently at 29% completion :((14/48)
- They are due SOON! (Tuesday 8/24) !!

Now to answer question - how to compute eigenvalues/vectors for linear transformations?

Thm: $T: V \to V$ be linear transformation. $\mathcal{B}$ a basis for V.

Then v is an eigenvector (with eigenvalue λ) for T if and only if

$[v]_{\mathcal{B}}$ is an eigenvector (with eigenvalue λ) for $A_{T,\mathcal{B}}$

ex) Find eigenvalues/vectors for

$T: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ by

$$T(a_0 + a_1 x + a_2 x^2) = (5a_0 + 2a_1 - 4a_2) + (6a_0 + 3a_1 - 5a_2)x + (10a_0 + 4a_1 - 8a_2)x^2$$

By thm: Form the matrix $A_{T,B}$ where $B=(1,x,x^2)$

$T(1)=5+6x+10x^2$
$T(x)=2+3x+4x^2$
$T(x^2)=-4-5x-8x^2$

$$\longrightarrow A_{T,B}=\begin{pmatrix}5 & 2 & -4\\ 6 & 3 & -5\\ 10 & 4 & -8\end{pmatrix}$$

$$\Rightarrow A_T-\lambda I=\begin{pmatrix}5-\lambda & 2 & -4\\ 6 & 3-\lambda & -5\\ 10 & 4 & -8-\lambda\end{pmatrix}$$

$$0=\det(A-\lambda I)=(5-\lambda)[(3-\lambda)(-8-\lambda)+20]-2[(6)(-8-\lambda)+50]+-4[24-10(3-\lambda)]$$

$$=(5-\lambda)[-4+5\lambda+\lambda^2]-2(-6\lambda+2)-4(-6+10\lambda)$$

$$=\cdots-\lambda^3+\lambda=0 \Rightarrow -\lambda(\lambda^2-1)=0 \quad \text{so} \quad \lambda=0,\pm 1$$

$\underline{\lambda_1 = 0}$: $\text{null}\begin{pmatrix} 5 & 2 & -4 \\ 6 & 3 & -5 \\ 4 & 0 & 4 & -8 \end{pmatrix} \longrightarrow \text{null}\begin{pmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \text{null } A = \text{span}\begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$

So $v = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$ is an eigenvector for $A_{T,B}$

So if $[w]_B = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$ then

$$\boxed{w_1 = \frac{2}{3} + \frac{1}{3}x + x^2}$$

$\Rightarrow$ Thm tells us that $\frac{2}{3} + \frac{1}{3}x + x^2$ is eigenvector for T with eigenvalue $\lambda = 0$

Check: $T\left(\frac{2}{3} + \frac{1}{3}x + x^2\right) = \left(5\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right) - 4\right) +$

$\left(6\left(\frac{2}{3}\right) + 3\left(\frac{1}{3}\right) - 5\right)x +$

$\left(10\left(\frac{2}{3}\right) + 4\left(\frac{1}{3}\right) - 8\right)x^2$

$= 0 + 0x + 0x^2$ ✓

$= 0$

$\underline{\lambda_2 = 1}$: $A - \mathbb{1}_3 = \begin{pmatrix} 4 & 2 & -4 \\ 6 & 2 & -5 \\ 10 & 4 & -9 \end{pmatrix}$

$\Rightarrow \text{null}(A - \mathbb{1}) = \text{null} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \text{null}(A - \mathbb{1}) = \text{span} \underline{\begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}}$. So $[w]_B = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$

Thm tells us that

$\underline{w_2 = \frac{1}{2} + x + x^2}$ is eigenvector of T

with eigenvalue $\lambda = 1$

$$T\left(\tfrac{1}{2}+x+x^2\right) = \left(5\left(\tfrac{1}{2}\right)+2-4\right) + \left(6\left(\tfrac{1}{2}\right)+3-5\right)x + \left(10\left(\tfrac{1}{2}\right)+4-8\right)x^2$$

$$= \frac{1}{2} + x + x^2 \quad \checkmark$$

$\underline{\lambda_3 = -1}$:

$$\begin{pmatrix} 6 & 2 & -4 \\ 6 & 4 & -5 \\ 10 & 4 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{null}\left(A + I\right) = \text{null}\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

So w is an eigenvector with $[w_3]_B = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$

$$\boxed{w_3 = \frac{1}{2} + \frac{1}{2}x + x^2}$$ with eigenvalue $\lambda = -1$

$$T\left(\frac{1}{2} + \frac{1}{2}x + x^2\right) = \left(5\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{2}\right) - 4\right) + \left(6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) - 5\right)x$$
$$+ \left(10\left(\tfrac{1}{2}\right) + 4\left(\tfrac{1}{2}\right) - 8\right)x^2$$
$$= -\frac{1}{2} - \frac{1}{2}x - x^2 = -1\left(\frac{1}{2} + \frac{1}{2}x + x^2\right) \checkmark$$

Fact: Eigenvectors coming from distinct eigenvalues are LI!

Then in this case $B' = (w_1, w_2, w_3)$ are a basis for $\mathbb{R}_2[x]$ (since they are LI). The matrix of T wrt this basis is

$$A_{T,B'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$T(w_1) = 0$

$T(w_2) = w_2$

$T(w_3) = -w_3$

Q: Is T an isomorphism? No, x_1 free variable. T not injective.

$\rightarrow$ this is the diagnolization of $A_{T,B}$

Ex 2) $T: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ with "standard" matrix

$$A_{T,\mathcal{B}} = \begin{pmatrix} 7 & 0 & 12 \\ -12 & -5 & -12 \\ -8 & 0 & -13 \end{pmatrix} \quad (\mathcal{B} = (1, x, x^2))$$

$\Rightarrow$ 2nd column tells us $\underline{T(x) = 0 + (-5)x + 0x^2 = -5x}$

So $\underline{x}$ is eigenvector with eigenvalue $\underline{-5}$

Long way to find any other eigenvalues

$$\det \begin{pmatrix} 7-\lambda & 0 & 12 \\ -12 & -5-\lambda & -12 \\ -8 & 0 & -13-\lambda \end{pmatrix}$$

$$= (7-\lambda)\left[(-5-\lambda)(-13-\lambda)\right] + 12\left(8(-5-\lambda)\right)$$

$$= -\lambda^3 - 84\lambda + 445$$

$$= (\lambda+5)^2(\lambda+1) \implies \lambda = \underline{-1} \text{ and } \lambda = \underline{-5}$$

$\underline{\lambda = -1}$: $\text{null}(A+1) = \begin{pmatrix} 8 & 0 & 12 \\ -12 & -4 & -12 \\ -8 & 0 & -12 \end{pmatrix} \xrightarrow{\text{do work}} \text{null}\begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$

so $\text{null}(A+1) = \text{Span}\begin{pmatrix} -3/2 \\ 3/2 \\ 1 \end{pmatrix}$ so

w is an eigenvector with $[w]_B = \begin{pmatrix} -3/2 \\ 3/2 \\ 1 \end{pmatrix}$

So $w = -\frac{3}{2} + \frac{3}{2}x + x^2$ with eigenvalue -1

$\lambda_i = -5$: $\mathrm{null}(A+5I) = \begin{pmatrix} 12 & 0 & 12 \\ -12 & 0 & -12 \\ -8 & 0 & 8 \end{pmatrix} \longrightarrow \mathrm{null}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathrm{null}(A+5I) = \mathrm{span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

so w is an eigenvector with $[w]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

so $w = x$ with eigenvalue $\lambda = -5$

Only have 2 eigenvectors for T

- we cannot diagonalize this matrix $A_{T,B}$

2 fun facts

A $n \times n$ matrix

1) $tr(A)$ = sum of the eigenvalues "trace of A"

2) $det(A)$ = product of the eigenvalues.

ex) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & -36 \end{pmatrix}$

What are the eigenvalues / vectors of A?

$\lambda_1 = 1 \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_1$

$\lambda_2 = 12 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = e_2$

$\lambda_3 = 12 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = e_3$

$\lambda_4 = -36 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_4$